國立中央大學 112 學年度碩士班考試入學試題

所別： 數學系碩士班 共 1 頁 第 1 頁

科目： 線性代數

注意事項
- 作答於答案卷並標明題號，作答時務必詳列解題過程，否則不予計分。
- 本試題卷共有 **9** 個問題, 總計 **100** 分。

1. (10 %) Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix.
(1) Prove that $\text{rank}(AB) \leq \text{rank}(B)$. (2) Show that $A^T A$ has the same nullspace (or kernel) as A.

2. (10 %) Let A be an $n \times n$ matrix such that $A^2 = A$.
(1) Find all possible eigenvalues of A. (2) Prove that A is diagonalizable.

3. (10 %) Is the following matrix A diagonalizable? Justify your answer.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

4. (10 %) Let

$$A = \begin{bmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{bmatrix}$$

(a) Find all values of t, for which A has 3 nonzero eigenvalues.
(b) Find all values of t, for which A has 3 positive eigenvalues.

5. (10 %) Compute $\lim_{n\to\infty} P^n$, where

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}$$

6. (10 %) Suppose B is a square matrix that satisfies $2B^2 + B - I = \mathbf{0}$, where I is the identity matrix and $\mathbf{0}$ is the zero matrix.
(1) Show that the inverse of B exists. (2) Find all possible eigenvalues of B^{-1}.

7. (10 %) Suppose that A and B are $n \times n$ matrices that commute, that is $AB = BA$. Show that there **exists** an eigenvector v of A which is also an eigenvector for B.

8. (15 %) Consider the following matrix A. Find real orthogonal matrices U, V and a 2×3 matrix Σ in the singular value decomposition $A = U\Sigma V^T$ for A, such that Σ has the following form:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & ? & ? \\ ? & ? & ? \end{bmatrix}$$

9. (15 %) Find all possible Jordan canonical forms of an $n \times n$ complex matrix A which satisfies the following six conditions simultaneously, where I is the $n \times n$ identity matrix:

(1) $(A - 2I)^2(A - 3I)^3$ is the $n \times n$ zero matrix.
(2) The rank of $(A - 2I)(A - 3I)$ is 3.
(3) The rank of $(A - 2I)^2(A - 3I)$ is 2.
(4) The rank of $(A - 2I)(A - 3I)^2$ is 2.
(5) The rank of $A - 2I$ is 6.
(6) The trace of A is 2023