國立中央大學105學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)
數學系 碩士班 應用數學組(一般生)
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科目： 線性代數

本科考試禁用計算器

＊請在答案卷(卡)內作答

Instructions: Do all 7 problems. Show your work. The notations: $\mathbb{R}$, is the set of all real numbers, $P_n(\mathbb{R})$ is the set of all polynomials of degree n with real valued coefficients.

1. (a) For any subsets U, W of vector space V, define
$$U + W = \{u + w | u \in U, w \in W\}.$$
Prove that if U, W are subspaces of V, then $U + W$ is also a subspace of V. (10 pts)

   (b) Let V be a finite dimensional inner product space, and U and W be two subspaces of V. If $\dim U < \dim W$, show that there exists a nonzero vector w in W such that w is orthogonal to all vectors in U. (10 pts)

2. Suppose A is a $n \times n$ square matrix with $A^k = 0$ for some positive integer k and x is a $n \times 1$ matrix such that $A^{k-1}x \neq 0$. Show that $\{x, Ax, \cdots, A^{k-1}x\}$ is linearly independent. (10 pts)

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by
$$T(x, y, z) = (x + 2y + z, y + z, -x + 3y + 4z).$$
Find a basis for the null space of T, the dimension of the null space, $dimN(T)$ and the dimension of the range, $dimR(T)$. (10 pts)

4. Let A be an invertible $n \times n$ matrix with real coefficients and $\mathrm{adj}(A)$ its classical adjoint.
   (a) What is the relation between A^{-1} and $\mathrm{adj}(A)$? (2 pts)
   (b) Show that if $\mathrm{adj}(A)$ is symmetric, then A is also symmetric. (4 pts)
   (c) Show that if $\mathrm{adj}(A) = \mathrm{I}, n \geq 2$, then $A = \mathrm{I}$ or $A = -\mathrm{I}$. (4 pts)

5. Let A be the 3×3 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 2$ for which $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ are the corresponding eigenvectors.
   (a) Find A. (10 pts)
   (b) Calculate $(A + \mathrm{I})^{100}$. (5 pts)

6. Let $V = P_n(\mathbb{R})$ with the inner product $< f(x), g(x) >= \int_{-1}^{1} f(t)g(t)\frac{1}{\sqrt{1-t^2}}dt$, and consider the subspace $P_2(\mathbb{R})$ with the standard ordered basis $\beta = \{1, x, x^2\}$. Use the Gram-Schmidt process to replace β by an orthonormal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$. (10 pts)

7. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$.
   (a) Find the minimal polynomial of A. (10 pts)
   (b) Explain why A is diagonalizable or not diagonalizable. (5 pts)
   (c) Find a Jordan canonical form J of A. (5 pts)
   (d) Find a matrix Q such that $Q^{-1}AQ = J$. (5 pts)