1. Let $S \subset [0,1]$ consist of all infinite decimal expansions $x = 0.a_1 a_2 a_3 \cdots$ where all but finitely many digits are 1 or 2. Find and prove $\sup(S)$. (10%)

2. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$. Show that $B = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid \sqrt{x_1^2 + \cdots + x_n^2} < 1\}$ is open set. (10%)

3. Let $f : [0,1] \mapsto [0,1]$ be continuous function. Prove that $\{x \in [0,1] \mid f(x) \in [1/3, 1/2]\}$ is compact set. (10%)

4. Use $\epsilon - \delta$ definition of limit to prove that $\lim_{x \to 0} \sqrt{4 - x} = 2$. (10%)

5. Let $f(x) = \sqrt{x}$ on $[0.1, \infty)$. Is f uniformly continuous? Give your reason. (10%)

6. Let f be function defined on $[0,1]$. Give an example to show that f is not Riemann integrable but $|f|$ is Riemann integrable. (10%)

7. Let $f_n(x) = x^n, 0 \leq x \leq 1$. Does f_n converge uniformly? What if $0 \leq x < 1$? Please justify your answers. (10%)

8. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x) x^n dx = 0 \qquad n = 0, 1, 2, 3, \cdots,$$

prove that $f(x) = 0$ on $[0,1]$. (10%)

9. Show that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous. (10%)

10. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$

(a) Show that both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin. (5%)
(b) Prove that f is not differential at the origin. (5%)