# 國立中央大學104學年度碩士班考試入學試題

所別：數學系碩士班 一般組(一般生)　科目：線性代數　共 1 頁 第 1 頁
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本科考試禁用計算器　　＊請在答案卷（卡）內作答

參考用

Instructions: Do all 5 problems. Show your work. The notations: $\mathbb{R}$, is the set of all real numbers, $P_n(\mathbb{R})$ is the set of all polynomials of degree $n$ with real valued coefficients, and $M_{n\times n}(\mathbb{R})$ is the set of all real-valued $n$ by $n$ matrices.

1. (a) Show that the set of solution to the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases}$$

is a subspace of $\mathbb{R}^3$ (6 pts). (b) Find a basis for the space and the dimension of the subspace. (9 pts)

2. Give the definition of symmetric positive definiteness of a matrix $A \in M_{n\times n}(\mathbb{R})$. (5 pts) Show that if $A \in M_{n\times n}(\mathbb{R})$ is symmetric and positive definite, all eigenvalues of $A$ is real and positive (10 pts).

3. Consider the matrix $A \in M_{3\times 3}(\mathbb{R})$ given by

$$\begin{pmatrix} -2 & 0 & 6 \\ -1 & 1 & 2 \\ -2 & 0 & 5 \end{pmatrix}$$

   (a) Find all eigenvalues of $A$ and corresponding vectors (10 pts)

   (b) Test $A$ for diagonalizability, and if $A$ is diagonalizable, find an invertible matrix $Q$ and a diagonal matrix $D = (d_{ii})$, where $1 \leq i \leq 3$ with $d_{11} \geq d_{22} \geq d_{33}$ such that $D = Q^{-1}AQ$. (10 pts)

   (c) Compute $A^n$, where $n \in \mathbb{N}$. Simplify your formula as much as possible. (10 pts)

   (d) Find a matrix $B \in M_{3\times 3}(\mathbb{R})$ such that $B = A^2$. (10 pts)

4. Determine whether each of the following is an inner product on given vector space. If not, please provide the reasons. (10 pts)

   (a) $< (a,b),(c,d) >= ac - bd$ in $\mathbb{R}^2$

   (b) $< A, B >= trace(A + B)$ on $M_{2\times 2}(\mathbb{R})$.

   (c) $< p, q >= \int_0^1 p(x)q(x)\,dx$ on $P_2(\mathbb{R})$

5. Consider the matrix $A \in M_{3\times 3}(\mathbb{R})$ defined as $A = I - 2uu^T$, where $I$ an identity matrix and $u$ is a unit column vector.

   (a) Show that $A$ is a symmetric and orthogonal matrix (10 pts).

   (b) Let $u = (1,1,1)^T/\sqrt{3}$. Solve the linear system, $Ax = b$, where $b = (1,0,0)^T$ (10 pts).