Instructions: Answer the following questions. Make and state your own assumptions for questions where the information is not sufficient for you to solve them. For example, if you need the corresponding p-value of a normally distributed random variable evaluated at 2.5, you may indicate the value as, say, Pr(x ≥ 2.5), where x ~ N(0, 1).

PART I

1. (25 points) Given $f(x; \theta) = 1 + \theta^2 e^{-\theta x}$, $0 \leq x \leq 1$, $0 \leq \theta \leq \sqrt{2}$.

(a) (10 points) Calculate the mean and variance of x?

(b) (15 points) If you are given only one observation X, how will you test the null hypothesis $H_0: \theta = 0$ against the alternative hypothesis $H_1: \theta > 0$? Write down the distribution of the statistic you use. Derive the "critical region" for this test, given the type I error of 10%.

2. (25 points) Suppose you are given a sample of observations which are independently and identically distributed, i.e., $x_1, \ldots, x_n \sim iid \ N(\mu, \sigma^2)$. The sample mean and variance are: $\bar{x} = 0.0335, s^2 = 0.025$. Now two students would like to make some statistical inference about the sampling distribution of the sample mean, but they do not agree with each other.

(a) Student A says that the sampling distribution of $\bar{x}$ is normal with mean 0.0335 and variance 0.025, i.e., $\bar{x} \sim N(0.0335, 0.025)$.

(b) Student B does not agree with Student A. She thinks that one has to test if the mean is significantly different from zero first. So she tests the null hypothesis $H_0: \mu = 0$, and could not reject it. Therefore she concludes the sampling distribution should be: $\bar{x} \sim \chi^2(0, 0.025)$.

Who is correct? Are there any problems with their statements?

PART II

3. (1)(5 points)

Consider the least squares residuals, given by $y_i - \hat{y}_i (i=1,2,\ldots,n)$.

Show that $\sum_{i=1}^{n} \frac{y_i - \hat{y}_i}{n} = \bar{y}$.

(2)(5 points)

Show that, for the simple linear regression model,

$\sum_{i=1}^{n} (y_i - \hat{y}_i) = 0$.

(3)(7 points)

The estimator of the error variance, $\sigma^2$, is given by

$S^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{n-2}$

show that the estimator is unbiased, prove $E(S^2) = \sigma^2$.

(4)(8 points)

For the simple linear regression model, show that

$b_1 = \frac{S_{xy}}{S_{xx}}$ and $\hat{y} = \frac{\sum_{i=1}^{n} y_i}{n}$

have zero covariance.

4. (1)(8 points)

Let X be a random variable having an exponential density with parameter $\lambda$.

Find the density of $Y = X^{1/\beta}$, where $\beta \neq 0$.

(2)(8 points)

Let X and Y be independent random variables each having an exponential distribution with parameter $\lambda$. Find the distribution of X+Y.

(3)(9 points)

Let X and Y be independent and uniformly distributed over (0,1). Find the density of X+Y.