(30%) 1. Define a function of $y$ as follows:

$$f(y) = \begin{cases} p\eta_1 e^{-\eta_1 y} & y \geq 0, \\ q\eta_2 e^{\eta_2 y} & y < 0, \end{cases}$$

where $p, q \geq 0$, $p + q = 1$, $\eta_1, \eta_2 > 0$. Please answer the following questions.

(a) Compute the integral: $\int_{-\infty}^{\infty} y f(y) dy$.

(b) Define the cumulative function $F(y) = \int_{-\infty}^{y} f(t) dt$.

(20%) 2. Let's define a function of $x$ as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}, \quad -\infty < x < \infty,$$

where $\sigma \in \mathbb{R}^+$, and then we have the following equation:

$$\int_{-\infty}^{\infty} \text{Max}(e^x - K, 0) f(x) dx = e^{\frac{x^2}{2}} \Phi(D) - K \Phi(D - \sigma),$$

where $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$, $-\infty < a < \infty$, and $K$ is a constant. Please compute $D$.

(15%) 3. Please compute $\frac{\partial C(\sigma)}{\partial \sigma}$, where $C(\sigma)$ is defined as follows:

$$C(\sigma) = S \phi(d_1) - K e^{-\tau T} \phi(d_2),$$

where $d_1 = \ln\left( \frac{a}{b} \right) (\tau + \frac{r}{2}) \tau$ and $d_2 = d_1 - \sigma \sqrt{T}$, and $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$.

(15%) 4. Solve the equation: $xy' = x^2 + Cx^4$, $x > 0$, given the initial condition $y(1) = 2$.

(20%) 5. If $X$ has the probability density function $f(x) = \frac{1}{4}$, $-1 < x < 3$, zero elsewhere, find the probability function of $Y = X^2$. 