Problem 1

(3%) (a) Given the function

$$f(x, y, z; a) = [x^2+y^2+(z-\tfrac{1}{2}a)^2]^{-1/2} - [x^2+y^2+(z+\tfrac{1}{2}a)^2]^{-1/2}$$

for the potential of a pair of oppositely-signed monopoles, find its gradient vector.

(3%) (b) Use Taylor expansion of the above function to find the potential function

$$F = \frac{a}{x^2+y^2+z^2}$$

for a dipole with a dipole moment a.

(3%) (c) Calculate the gradient of the dipole's potential function in rectangular coordinates.

(3%) (d) Calculate the divergence of the gradient vector.

(3%) (e) Writing the dipole's potential function as

$$F = \frac{a}{r^2}$$

in spherical coordinates, calculate its gradient in spherical coordinates.

Problem 2

(10%) Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$.

Problem 3

(10%) Solve the system of differential equations

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = 2u - v + 4$$

subject to the initial values of u equal to 1 and v equal to 0.

Problem 4

(5%) (a) Show that t^2 and $\frac{1}{t}$ are two solutions to the differential equation $t^2\frac{d^2u}{dt^2} - 2u = 0$.

(10%) (b) Solve the inhomogeneous differential equation $t^2\frac{d^2u}{dt^2} - 2u = 3t^3$.

Problem 5

(10%) Solve the first-order differential equatioin

$$\frac{du}{dt} - 2u = t^2.$$

Problem 6

(10%) Solve the second-order differential equatioin

$$\frac{d^2u}{dt^2} - \frac{du}{dt} - 2u = 3e^{-t} + 10\sin t - 4t.$$

Problem 7

(5%) (a) Write down the Euler formula and prove it by infinite series.

(10%) (b) Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{1}{2^n}\cos nx.$$

Problem 8

(10%) (a) Calculate the Fourier series for the function with period 2π

$$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi<x<0 \\ \frac{\pi}{4}, & 0<x<\pi \end{cases}$$

(5%) (b) Evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$.

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