1. (a) Solve the initial value problem

\[ ay'' + by = 0, \quad y(0) = 0, \quad y'(0) = 1. \]

where \( a \) and \( b \) are constants, but \( a \neq 0 \).

(b) Find a basis of solution of the differential equation. (Show the details of your work.)

\[ x'y'' + 3xy' + y = 0 \]

(c) Find the Laplace transforms of the following function. (Show the details of your work.)

\[ f(t) \]

2. (a) Evaluate \( \oint_C \frac{e^z}{(z-1)(z+4)} \, dz \), where \( C \) is the circle \( |z| = 3 \) described in the positive direction.

(b) Evaluate \( \oint_C e^z \sin(1/z) \, dz \), where \( C \) is the circle \( |z| = 1 \) described in the positive direction.

(c) Evaluate \( \int_0^\pi \cos 2\theta \, d\theta \).

3. (a) Find the similarity transformation \( A = PAP^{-1} \), where \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) and \( A \) is a diagonal matrix.

(b) Consider a system of differential equations \( \frac{dy}{dx} = Ay \) subject to the initial condition \( y(0) = b \), where the matrix \( A \) is given as above. \( y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). We can solve this problem by taking the iterative procedure:

\[
y^{(0)} = b,
\]

\[
y^{(1)} = b + \int_A y^{(0)} d\xi = b + xAb,
\]

\[
y^{(2)} = b + \int_A y^{(1)} d\xi = b + xAb + \frac{(xA)^2}{2!} b,
\]

\[
y^{(n)} = \left[ 1 + xA + \frac{(xA)^2}{2!} + \ldots + \frac{(xA)^n}{n!} \right] b,
\]

and \( y^{(n)} \rightarrow y \) as \( n \rightarrow \infty \). Obtain \( y_1 \) and \( y_2 \) by the iteration method and the similarity transformation you have got. Show the details of your work. (Think about the Taylor series expansion for \( e^z \) about \( z = 0 \).)

4. By the method of separation of variables, find the solution \( u(x,y) \) of the Poisson equation

\[ u_{xx} + u_{yy} = \cos(\pi y), \]

in the semi-infinite strip \( 0 \leq x < \infty, 0 \leq y \leq 1 \), such that

\[ u(0,y) = y, \quad u_x(x,0) = u_x(x,1) = 0. \]