國立中央大學106學年度碩士班考試入學試題

所別：通訊工程學系碩士班 不分組(一般生)

科目：工程數學(線性代數、機率)

本科考試禁用計算器 *請在答案卷 內作答

須有計算過程

1. For $\mathbf{A} = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & 4 \end{bmatrix}$ with one known eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

   find (a) the eigenvalue λ_1 corresponding to eigenvector $\vec{v}_1$; (5%)

   (b) the eigenvector $\vec{v}_2$ corresponding to eigenvalue $\lambda_2 = 3$; (5%)

   (c) the least squares solution (i.e., $\vec{x}$) of $\min_{\vec{x}} \left\| \vec{b} - \overline{\mathbf{A}} \cdot \vec{x} \right\|^2$ when $\overline{\mathbf{A}} = [\vec{a}_1 \quad \vec{a}_2]$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; ($\|\vec{v}\|^2 = \vec{v}^T \cdot \vec{v}$) (5%)

   (d) a vector $\vec{b} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ such that $\min_{\vec{x}} \left\| \vec{b} - \overline{\mathbf{A}} \cdot \vec{x} \right\|^2 = \left\| \vec{b} \right\|^2$; (5%)

   (e) the value (x, y) in $\mathbf{A}^2 \cdot (2 \cdot \vec{v}_1 + \vec{v}_2) = x \cdot \vec{v}_1 + y \cdot \vec{v}_2$; (5%);

   (f) the value $M = \max\{\vec{x}^T \cdot \mathbf{A} \cdot \vec{x} : \ \|\vec{x}\|^2 = 2\}$ (5%);

參考用

2. Let H be the set (inner product space) of all waveforms described by $s(t) = a_1 \cdot v_1(t) + a_2 \cdot v_2(t) + a_3 \cdot v_3(t)$, with $a_i \in R$, $v_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$,

   $v_2(t) = \begin{cases} -1, & 0 \le t \le 0.5 \\ 1, & 0.5 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$, $v_3(t) = \begin{cases} 1, & 0 \le t \le 0.5 \\ 0, & \text{otherwise} \end{cases}$ and the inner product defined by $s_1(t) \bullet s_2(t) = \int_0^1 s_1(t) \cdot s_2(t) dt$.

   Find (a) $\|v_3(t)\|^2 = v_3(t) \bullet v_3(t)$; (5%)

   (b) $v_2(t) \bullet v_3(t)$; (5%)

   (c) the value (a, b) such that $\hat{v}_3(t) = v_3(t) - a \cdot v_1(t) - b \cdot v_2(t)$, $v_1(t) \bullet \hat{v}_3(t) = 0$ and $v_2(t) \bullet \hat{v}_3(t) = 0$ (Gram-Schmidt Process with the defined inner product); (5%)

注意：背面有試題

(d) the least squares solution of $\min_{a_1,a_3} \left\| w(t) - a_1 \cdot v_1(t) - a_3 \cdot v_3(t) \right\|^2$ when

$$w(t) = \begin{cases} 1, & 0 \le t \le 0.5 \\ 4, & 0.5 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$ (hint: use orthogonal projection). (5%)

3. For two identical independent distribution (i.i.d.) random variables X and Y with a marginal probability density function (pdf) $f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$, find

   (a) $\Pr(X > 0.8)$, i.e., the probability of the event $(X > 0.8)$; (5%)

   (b) $\Pr(0.3 < X < 0.8, Y < 0.8)$; (5%)

   (c) $\Pr(X + Y < 0.8)$; (5%)

   (d) the pdf of $Z = X^2$; (5%)

   (e) the pdf of $Z = X + 2 \cdot Y$; (5%)

   (f) $E\{X^2 \cdot Y\}$, $E\{\ \}$: ensemble average (expectation) (5%)

4. For a Gaussian random variable X with known $m_1 = E\{X\}$ and $m_2 = E\{X^2\}$, find (a) the pdf (equation $f_X(x)$) of X; (5%)

   (b) $\Pr(X > A)$ in terms of Q-function ($Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp\left(\frac{-x^2}{2}\right) dx$); (5%)

   (c) $\mathrm{var}\{2X + A\}$, ($\mathrm{var}\{X\} = E\{X^2\} - (E\{X\})^2$). (5%)

5. For two correlated random variables X and Y with known $m_X = E\{X\}$, $\sigma_X^2 = \mathrm{var}\{X\}$, $m_Y = E\{Y\}$, $\sigma_Y^2 = \mathrm{var}\{Y\}$ and $c_{X,Y} = E\{X \cdot Y\}$, find the value a such that $\min_a E\{(Y - a \cdot X)^2\}$. (5%)