Problem 1 (15%)

Please answer the following questions.

a) (5%) If $D = P^{-1}AP$ and D is diagonal, prove that $A^k = PD^kP^{-1}$.

b) (10%) Find A^{58}, where $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$.

Problem 2 (15%)

Prove that: If $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are linearly independent.

Problem 3 (15%)

Find the rank and nullity of the matrix A, where

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}.$$

Problem 4 (20%)

Find the response of the following undamped mass-spring system.

$$y'' - y = 10\delta(t - \frac{1}{2}) - 100\delta(t-1), \ y(0) = 10, \ y'(0) = 1$$

Problem 5 (20%)

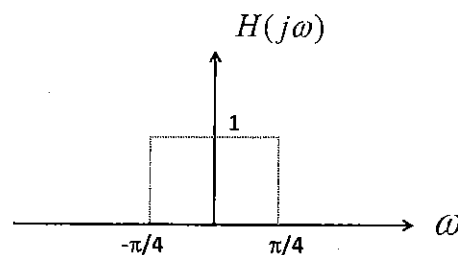Find the solution of the following equation. (Please show the details of your solution)

$$y(t) * t^{-1/2} = 1 + t,$$

where * is the continuous-time convolution operator.

(hint: Gamma function $\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt, \ \Gamma(1/2) = \sqrt{\pi}$)

Problem 6 (15%)

For a continuous-time low-pass filter system with its Fourier response $H(j\omega)$

Please answer the following questions:

a) (10%) Please use continuous-time inverse Fourier transform to find the impulse response $h(t)$ of the low-pass filter system.

b) (5%) For an input signal $x(t) = \sin(\frac{\pi}{6} t) + \cos(\frac{\pi}{2} t)$, please find the output of this low-pass filter.