一. (20%) Let $X(t)$ be a zero-mean wide-sense stationary random process with autocorrelation function $E\{X(t)X(\tau)\} = R_X(\tau)$ and power spectral density (PSD) $S_X(f)$.

(一) (4%) Let $Y(t) = X(t)\cos(2\pi f_c t + \theta)$, where $f_c$ is a known constant, and $\theta$ is a random variable uniformly distributed over $(\pi, -\pi)$ that is independent of $X(t)$. Find the PSD of $Y(t)$.

(二) (5%) Following (一), if $X(t)$ is a Gaussian random process, is $Y(t)$ also a Gaussian random process? Why?

(三) (6%) If $X(t)$ is passed through a finite-time integrator to obtain $Z(t) = \frac{1}{T} \int_{t-T}^{t} X(\tau)d\tau$, find the PSD of $Z(t)$.

(四) (5%) Following (三), if $X(t)$ is a white noise process with PSD $S_X(f) = N_0/2$, find the average output power $E(\|Z(t)\|^2)$, where $E(\cdot)$ is the expectation operator.

二. (20%) Assume a voltage-controlled oscillator (VCO) is used as an FM modulator, whose voltage to frequency transfer coefficient is $k$ (Hz/V). Let $m(t)$ be the input of the VCO and $s(t)$ be the modulated FM signal, given as

$$m(t) = A \cos(2\pi f_c t) + B \cos(2\pi f_s t)$$
$$s(t) = \cos(2\pi f_c t + \theta(t))$$

where $f_c$ is the carrier frequency and $f_s > f_c$.

(一) (6%) Derive $s(t)$.

(二) (7%) If $A = 3V$, $B = 5V$, $k = 15KHz/V$ and $f_c = 2KHz$, $f_s = 6KHz$, $f_c = 100MHz$, find the deviation ratio ($D$) of the FM signal.

(三) (7%) Assume the above FM signal passes through a band-pass filter (BPF) whose bandwidth is defined by the Carson's rule. After the BPF, the FM signal is amplified with a nonlinear amplifier. The input-output relationship of the amplifier is given as

$$y = a_0 x + a_1 x^3 + a_2 x^4$$

where $x$ is the amplifier input, $y$ is the output, and $a_0, a_1, a_2$ are non-zero constants. At the amplifier output, find the bandwidth of the FM signal at the carrier frequency $f_c$ and explain your answer.
三. (20%) Consider a system with two equally probable signals $s_1(t)$ and $s_2(t)$ given by

\[ s_1(t) = \begin{cases} A, & \text{for } 0 < t < T \\ 0, & \text{otherwise} \end{cases}, \quad s_2(t) = \begin{cases} -A, & \text{for } 0 < t < T \\ 0, & \text{otherwise} \end{cases}. \]

The received signal is defined as

\[ r(t) = s_i(t) * h(t) + n(t), \quad i = 1 \text{ or } 2, \]

where $*$ denotes linear convolution, $h(t)$ is the impulse response of a linear time-invariant channel, and $n(t)$ is white Gaussian noise of zero mean and two-sided power spectral density $S_n(f) = N_0/2$.

(一) (4%) Suppose $h(t) = \delta(t)$, where $\delta(t)$ denotes the ideal impulse function. Draw the block diagram of matched filter receiver and find its average probability of symbol error.

(二) (6%) Redo (一) if $h(t) = \delta(t) + \delta\left(t - \frac{T}{2}\right)$.

(三) (5%) Suppose $h(t) = \delta(t) + \beta\delta\left(t - \frac{T}{2}\right)$, where $\beta$ is a random variable with probabilities

\[ \text{Prob}(\beta = 1) = \text{Prob}(\beta = -1) = 1/2. \]

Find the average probability of symbol error if the optimum receiver in (一) is used for detection.

(四) (5%) Following (三), find the average probability of symbol error if the optimum receiver in (二) is used for detection.

四. (20%) Consider binary ($a_k = \pm 1$ equally likely) continuous-phase FSK modulation in an AWGN channel with two-sided noise power spectral density $S_n(f) = N_0/2$, assuming the bit duration is $T_b$ sec, the bit energy is $E_b$, the frequency spacing between the two transmitted tones is equal to $1/T_b$ Hz, and the carrier frequency is $f_0$ Hz.

(一) (6%) Derive and plot the power spectrum density of the FSK signal.

(二) (7%) Derive the optimum coherent receiver and draw its block diagram.

(三) (7%) Derive the optimum non-coherent receiver and draw its block diagram.
五．(20%)

(一) (6%) Consider a direct sequence spread spectrum (DSSS) QPSK system using only one spreading code \( c(i) \) with chip time \( T_c \) and a period of \( N \) chips. Suppose there also exists a single tone jamming interference in the channel. Plot the block diagrams of the transmitter and receiver, assuming that spreading is done after the QPSK modulation. Also, explain why a DSSS system has the anti-jamming capability.

(二) (7%) Consider a source symbol \( X \) taking values in the set \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) with probabilities 0.25, 0.1, 0.2, 0.1, 0.25, 0.05, 0.05, respectively. Please construct a ternary source code with symbol taking values from \( \{0, 1, 2\} \) for \( X \), which has the minimal average codeword length.

(三) (7%) Consider two binary random variables \( X \) and \( Y \) which take values from \( \{0, 1\} \) and have the following joint probabilities:

\[
\begin{align*}
\Pr(Y = 0, X = 0) &= 0.72, & \Pr(Y = 1, X = 0) &= 0.08 \\
\Pr(Y = 0, X = 1) &= 0.08, & \Pr(Y = 1, X = 1) &= 0.12
\end{align*}
\]

We can see that \( \Pr(Y = 0) = 0.8 \) and \( \Pr(Y = 1) = 0.2 \). Now Bob wants to estimate the value of \( Y \) from the observation of \( X \). Knowing that \( \Pr(Y = 0) > \Pr(Y = 1) \), Bob thus concludes that a simple way to estimate \( Y \) is to set \( Y = 0 \) all the time. Does Bob provide the best way to estimate \( Y \)? If yes, justify your answer. If not, find the best way to estimate \( Y \).