所別：數學系碩士班 甲組(一般生)　科目：高等微積分　共 2 頁 第 1 頁
數學系碩士班 甲組(在職生)
本科考試禁用計算器

*請在試卷答案卷（卡）內作答

參考用

Advanced Calculus

Master Program Entrance Exam for the Dept. of Math, Nat'l Central Univ.

Problem 1. Let $B = \left\{x \in \mathbb{R}^3 \,\middle|\, |x| < 1\right\}$ and $y = (0, 0, a)$ for some $a > 1$.

1. (5%) Show that the integral $\int_B \frac{1}{|y-x|}\,dx$ exists (note that this is a triple integral).

2. (10%) Find the value of the integral above.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|^\alpha \sin \dfrac{1}{|x|^\beta} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where α and β are positive numbers. Determine the ranges of α and β so that

1. (5%) $f(x)$ is uniformly continuous.

2. (5%) $f(x)$ is differentiable.

Problem 3. Complete the following.

1. (5%) State the definition of a compact set in a metric space.

2. (10%) Let a_k be a sequence in a metric space and $a_k \to a$. Show that the set $\{a_k\}_{k=1}^{\infty} \cup \{a\}$ is compact **based on the definition you stated.**

Problem 4. (10%) Suppose that $f_k : (a, b) \to \mathbb{R}$ is a sequence of differentiable functions such that $f_k(c)$ converges at one point $c \in (a, b)$, and f_k' converges uniformly on (a, b). Show that f_k converges uniformly.

注意：背面有試題

Problem 5. Let $A = [a, b] \times [a, b]$ be a bounded, closed square in $\mathbb{R}^2$, and $X = C([a, b], \mathbb{R})$ (be the space of continuous functions on $[a, b]$ equipped with uniform norm). Given $K : A \to \mathbb{R}$ a continuous function, define a map T on X by

$$T(f)(x) = \int_a^b K(x, y) f(y)\, dy \qquad f \in X.$$

1. (10%) Let B be a bounded subset of X; that is, there is a constant $M > 0$ such that $\max_{x \in [a,b]} |f(x)| \le M$ for all $f \in B$. Show that the set $T(B)$ is bounded in X and equi-continuous.

2. (6%) Show that T is a contraction mapping on X if $\max_{(x,y) \in A} |K(x, y)| < \dfrac{1}{b - a}$.

Problem 6. (10%) Define f on the square $Q = [0, 1] \times [0, 1]$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $f(x, y)$ is not Riemann integrable on Q by showing

$$\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx \ne \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy.$$

Problem 7. Prove or disprove the following statements.

1. (6%) Let $f : (0, 1) \to \mathbb{R}$ be a continuous function. If both limits $f(0+)$ and $f(1-)$ exist, then f is uniformly continuous on $(0, 1)$.

2. (6%) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of Riemann integrable functions, and f_n converges pointwise to f. Then f is Riemann integrable.

3. (6%) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable, and the Jacobian of f does not vanish everywhere; that is, $\det\big(Df(x)\big) \ne 0$ for all $x \in \mathbb{R}^2$. Then there exists an inverse function of f.

4. (6%) Let $\Omega \subset \mathbb{R}^n$,, and a be an interior point of Ω. Suppose that $f : \Omega \to \mathbb{R}$ is a function such that both partial derivatives $f_x(a)$ and $f_y(a)$ exist. If each directional derivative $D_u f(a)$ exists and $D_u f(a) = (\nabla f)(a) \cdot u$, then $f : \Omega \to \mathbb{R}$ is differentiable at a.