所別：數學系碩士班 甲組(一般生)　科目：線性代數　共 2 頁 第 1 頁
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*請在試卷答案卷（卡）內作答

參考用

1. **Dimension computation** (You need to explain your answers).

   (a) (10 %) Let $\ell_i(x_1,\dots,x_n)=\sum_{j=1}^{n} a_{ij}x_j$, where $a_{ij}\in\mathbb{R}$, $1\le i\le m, 1\le j\le n$ and let $W=\{(b_1,\dots,b_m)\in\mathbb{R}^m \mid \sum_{i=1}^{m} b_i\ell_i=0\}$ with $\ell_i=\ell_i(x_1,\dots,x_n)$. Let
   $$\mathrm{N}:=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \ell_i(x_1,\dots,x_n)=0, i=1,\dots,n\}.$$
   Express the dimension of N in terms of m,n and $\dim W$.

   (b) (10 %) Let V and W be finite dimensional vector space over a field F. Let S be a nonempty subset of V. Suppose that the subspace spanned by S has dimension d. Denote the space of linear transformations from V to W by $\mathcal{L}(V,W)$ and set
   $$U_S=\{T\in\mathcal{L}(V,W) \mid T(v)=\mathbf{0}_W \text{ for all } v\in S\}$$
   where $\mathbf{0}_W$ is the zero vector of W. Note that U_S is a subspace of $\mathcal{L}(V,W)$. Express the dimension of U_S in terms of $\dim V, \dim W$ and d.

2. Let V and W be finite dimensional vector spaces over a field F. For given linear transformations $f: V\to W$ and $g: W\to V$ let the linear transformation $\Phi: V\oplus W\to V\oplus W$ be defined by $\Phi((v,w))=(f(w),g(v))$ for all $(v,w)\in V\oplus W$.

   (a) (8 %) Show that Φ is one to one if and only if both f and g are one to one. In this case, is it true that both f and g are isomorphisms ?

   (b) (12 %) Prove or disprove that $\mathrm{rank}(\Phi)=\mathrm{rank}(f)+\mathrm{rank}(g)$ where for a linear transformation L its rank is denoted by $\mathrm{rank}(L)$.

3. Let the matrix
   $$A=\begin{pmatrix}1&0&0&0\\-9&-2&0&0\\0&0&1&0\\-1&0&0&1\end{pmatrix}.$$

   (a) (15 %) Find a Jordan canonical form J of A and an invertible matrix P such that $P^{-1}AP=J$.

   (b) (10 %) Consider $\mathcal{A}=\{A^n \mid n=0,1,2,\dots\}$ and let $\mathrm{Span}(\mathcal{A})$ be the subspace of $M_{n\times n}(\mathbb{R})$ spanned by $\mathcal{A}$ over the field of real numbers $\mathbb{R}$. Determine the dimension of $\mathrm{Span}(\mathcal{A})$.

4. (a) (5 %) Is it true that every finite dimensional inner product space has an orthogonal basis? Explain your answer.

   (b) (15 %) Let $\mathrm{N}\subset\mathbb{R}^5$ be the set of solutions to the following system of homogeneous linear equations.
   $$\begin{aligned} x_1 \quad + x_3 \quad\quad &= 0\\ x_1+x_2+x_3+x_4+x_5&=0\\ -x_1+2x_2-x_3+x_4 \quad &=0\\ 2x_1+3x_2+2x_3+2x_4+x_5&=0\end{aligned}$$
   Recall that the standard inner product (the dot product) on $\mathbb{R}^n$ is given by $\mathbf{u}\cdot\mathbf{v}=\sum_{i=1}^{n}u_iv_i$ for $\mathbf{u}=(u_1,\dots,u_n)$ and $\mathbf{v}=(v_1,\dots,v_n)$. Let $\mathrm{N}^\perp$ be the orthogonal complement of N in $\mathbb{R}^5$ equipped with the standard inner product. Give an orthogonal basis for $\mathrm{N}^\perp$.

注意：背面有試題

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5. (15 %) Fix a matrix $A \in M_{n\times n}(\mathbb{R})$ and let $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the linear operator defined by $T(M) = AM$ for all $M \in M_{n\times n}(\mathbb{R})$. Let $f(x)$ be the characteristic polynomial of A and let $f_T(x)$ be the characteristic polynomial of T. Prove that $f_T(x) = f(x)^n$.

參考用

注意：背面有試題